3.5

That Graph Looks a Little Sketchy

Building Cubic and Quartic Functions

LEARNING GOALS

In this lesson, you will:

- Construct cubic functions graphically from three linear functions.
- Construct cubic functions graphically from one quadratic and one linear function.
- Connect graphical behavior of a cubic function to key characteristics of its factors.
- Construct quartic polynomial functions.
- Determine the number of real and imaginary roots for a polynomial function based on its factors.

People in the world today use a lot of energy, much more than previous generations. Consider modern conveniences people in the U.S. have in public spaces such as heating, air conditioning, lights, and electronic devices. Also, consider food products, clothes, and other goods that often travel halfway around the world on planes, ships, or trucks before ending up in U.S. shopping malls. Quite a bit of energy goes into getting these resources to you. People also travel much more these days than ever before. You may ride a bus to school and shop at a mall; adults may commute 30+ miles to work; business people may fly across the country to attend a conference. Compare this lifestyle to how people lived throughout the vast majority of history. People generally grew their own food, traveled on foot, and made their own clothes and wares. Is our lifestyle sustainable? In other words, can we continue using this much energy forever?

We use approximately 1.2 trillion gallons of gasoline each year. We also use tremendous amounts of coal and natural gas. The world's current energy consumption is so large that the numbers are difficult to even comprehend. The unit of measure Cubic Mile of Oil was developed to help make sense of it. A CMO is literally the amount of energy released by burning a cubic mile of oil. To visualize a cubic mile, imagine a huge cube-shaped container with length, width, and height of approximately 18 football fields. The energy from burning three of these containers of oil is the amount of energy we currently use in just one year. At this rate of consumption our natural gas reserves will be gone by 2080. Coal reserves will run out by 2150.

It is hard to imagine people voluntarily returning to a world without the conveniences we have today. However, natural resources are limited. What options do we have if we want our children to live a life filled with the conveniences that we currently enjoy?

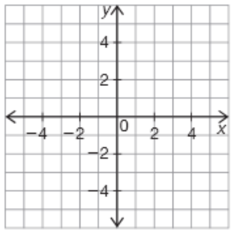
PROBLEM 1 They Don't Build Cubics Like They Used To!

So far in this chapter you've built a cubic function by multiplying three linear functions and by multiplying a linear function and a quadratic function. Let's explore how the properties of linear and quadratic functions determine the key characteristics of cubic functions.

1. Sketch a set of functions whose product builds a cubic function with the given characteristics. Explain your reasoning. Then list similarities and differences between your graphs and your classmates' graphs.

   a. zeros: $x = 0$, $x = 2$, and $x = -5$

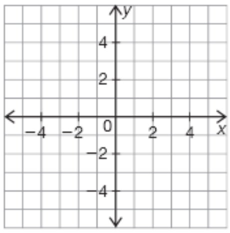
   Explanation:

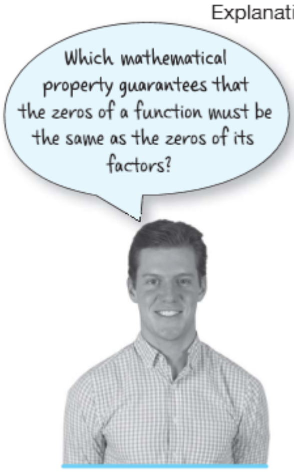


   Similarities/Differences:

b. zeros: $x = -3$, $x = 4$ (multiplicity 2)

Explanation:





Similarities/Differences:

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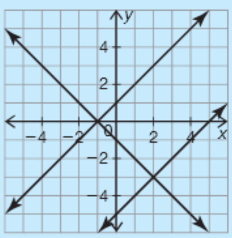
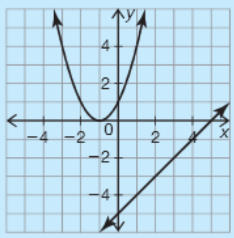
2. Alex and Derek disagree over which functions when multiplied together build a cubic function with zeros $x = 5$, $x = -1$ (multiplicity 2).

Alex



I sketched three linear functions, each with an x-intercept that matches the zero.

Derek



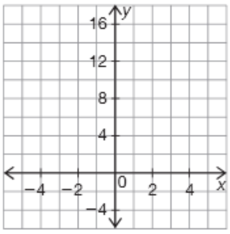
I sketched a parabola with vertex (–1, 0) and a line with x-intercept at (5, 0).

Who is correct? Explain your reasoning.

3. Sketch a set of functions whose product builds a cubic function with the given characteristics. Explain your reasoning. Then list similarities and differences between your graphs and your classmates' graphs.

a. two imaginary zeros and a real zero

Explanation:



Similarities/Differences:

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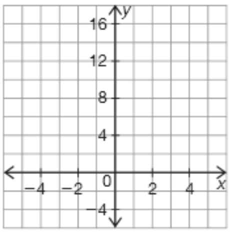
b. y-intercept of (0, 12)

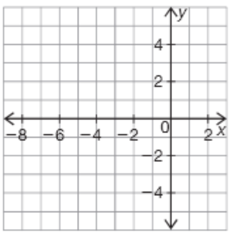
Explanation:



Similarities/Differences:

c. zero: $x = -4$ (multiplicity 3)

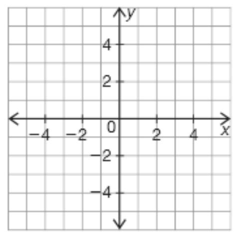
Explanation:

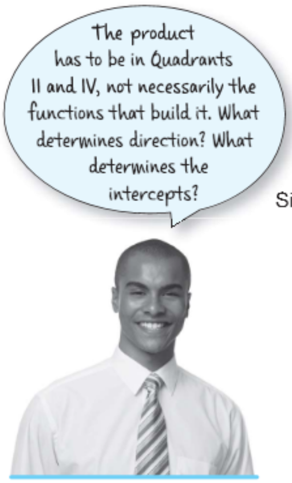


Similarities/Differences:

d. The cubic function is in Quadrants II and IV only.

Explanation:





Similarities/Differences:

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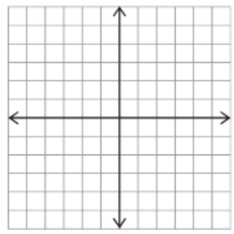
e. 3 imaginary roots



Explanation:

Similarities/Differences:

4. What are the possible combinations of real and imaginary roots that a cubic function can have? Explain your reasoning in terms of the functions that can build a cubic function.

5. Emily makes an observation about the number of imaginary zeros a cubic function may have.

Emily

A cubic function must have three zeros. I know this from the Fundamental Theorem. However, the number of real and imaginary zeros can vary. The function may have 0, 1, 2, or 3 imaginary zeros.

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Explain the error in Emily's reasoning.

6. Augie, Kathryn, and Chili each wrote a cubic function with zeros at $x = 3$, $x = 1$, and $x = -4$.

Augie

The cubic function $f(x) = (x - 3)(x - 1)(x + 4)$ has the three zeros given. I can verify this by solving the equations $x - 3 = 0$, $x - 1 = 0$, and $x + 4 = 0$.

Kathryn

The cubic function $g(x) = 5(x - 3)(x - 1)(x + 4)$ has the three zeros given.

Chili

The cubic function $j(x) = (2x - 6)(3x - 3)(x + 4)$ has the three zeros given.

a. How does multiplying by a constant affect the graph of the function?

b. Why do the zeros remain the same after multiplying by a constant?

c. How many different cubic functions can you write from a given set of zeros?

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7. Write two different cubic functions with the given characteristics.

a. zeros: $x = 2$, $x = 0$ and $x = -4$

b. zeros: $x = 0$, $x = 2i$, $x = -2i$

c. zeros: $x = 6$ (multiplicity 2) and $x = -5$

d. zeros: $x = 2$, $x = 3$, $x = 1$ and a y-intercept (0, 24)

e. the point (1, 12) lies on the graph of the function

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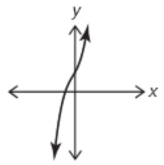
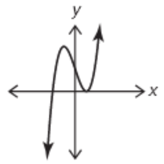
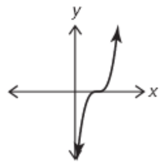
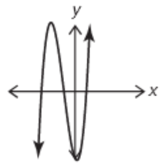
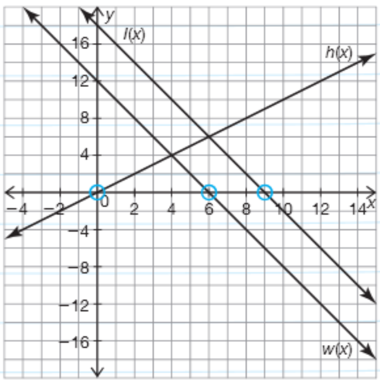
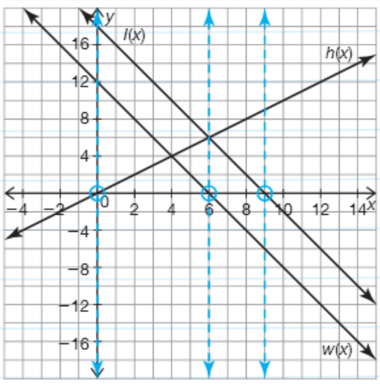
The factors and roots determine the general shape of a cubic function. The table summarizes all possible combinations of roots and factors for a cubic function.

Roots	Factors	Graph
1 real 2 imaginary	(linear factor) × (quadratic factor with 0 real roots)	
1 real (multiplicity 1) 1 real (multiplicity 2)	(linear factor) × (linear factor)^2	
1 real (multiplicity 3)	(linear factor)^3	
3 real distinct	(linear factor) × (linear factor) × (linear factor)	

PROBLEM 2 I Like My Cubics Built the Old-Fashioned Way

Recall that the volume function $V(x) = x(18 - 2x)(12 - 2x)$ from Plant-A-Seed was built by multiplying three linear functions representing length, width, and height. It was also built from a quadratic function representing the area of the base and a linear function representing the height. You can sketch the graph of a cubic function by determining the x-intercepts and the intervals for which the output values of the factors are positive or negative.
The Plant-A-Seed example is shown.

The linear functions that represent the length, width, and height of the planter boxes from Plant-A-Seed are shown on the graph.

Description	Graphical Display
Graph each factor as an individual function. • The x-intercepts for each function are circled.	
Draw dashed vertical lines through the x-intercepts. • The coordinate plane is now divided into 4 sections: $(-\infty, 0)$, $(0, 6)$, $(6, 9)$ and $(9, \infty)$.	

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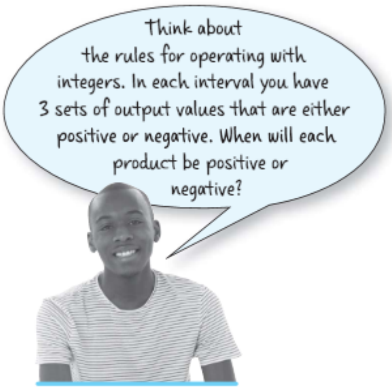

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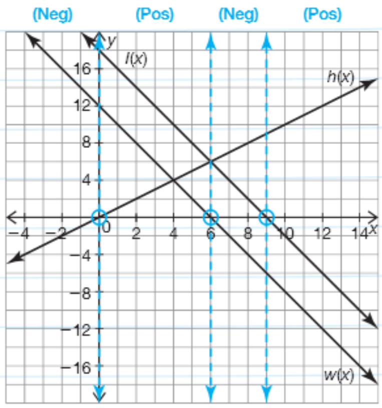
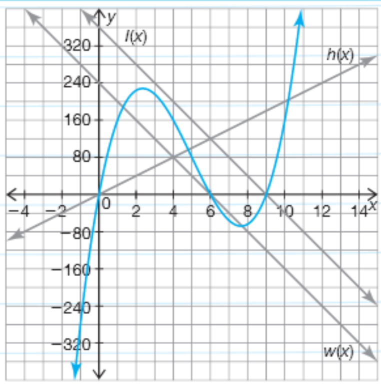
Determine whether the output values for each function in the interval are positive or negative.

- Values above the x-axis are positive.
- Values below the x-axis are negative.
- Determine the location of the cubic function by calculating whether the product of the factors is positive or negative over each interval.



- Use the x-intercepts and the sign of the output value over each interval to sketch the graph.
- The new function will cross the x-axis at each of the x-intercepts as the factors.
- The graph will increase or decrease depending on whether the output is positive or negative as it moves from one interval to the next.



1. Analyze the worked example.

   a. Given the three functions $l(x)$, $w(x)$, and $h(x)$, summarize how to determine when $V(x)$ lies above or below the x-axis.

   b. Why must the volume function intersect the x-axis at (0, 0), (6, 0), (9, 0)?

   c. Is it possible for a function to have a zero that is different from its factors? Explain your reasoning.

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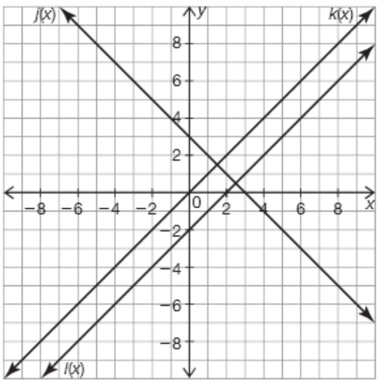
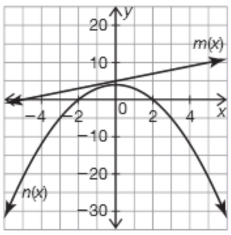
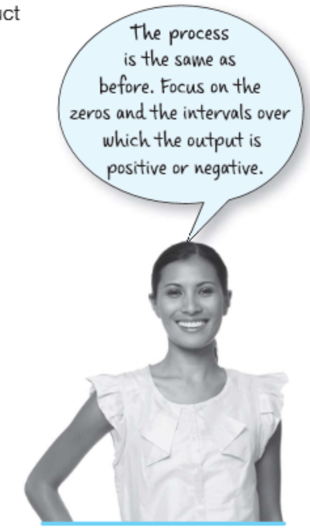
2. Sketch the graph of the cubic function that is the product of the 3 linear functions shown. Show all work and explain your reasoning.



3. Sketch the graph of the cubic function that is the product of the quadratic and linear functions shown. Show all work and explain your reasoning.





4. In Question 2 you graphically determined the product of the functions $f(x) = 3 - x$, $g(x) = x$ and $h(x) = x - 2$.

a. Determine the product of the functions algebraically.

b. Verify your sketch by graphing the product on a graphing calculator.

5. In Question 3 you graphically determined the product of the functions $j(x) = 4 - x^2$ and $k(x) = x + 5$

a. Determine the product of the functions algebraically.

b. Verify your sketch by graphing the product on a graphing calculator.

PROBLEM 3 Anyone Have Change for a Quartic?

In Problems 1 and 2, you determined that a cubic function has 3 zeros. The zeros may be real, imaginary, or have multiplicity depending on the key characteristics of the functions that built it. Similarly, the Fundamental Theorem of Algebra guarantees that a quartic function has 4 zeros. The key characteristics of the quartic function also vary depending on the functions that built it.

1. Analyze the linear, quadratic, and cubic functions that are shown.

$f(x) = x$	$g(x) = -x + 2$	$m(x) = x^2 - 2x - 5$
$p(x) = x^2 + 4$	$r(x) = (x + 2)^2$	$w(x) = x^3$

a. List the number and type of zeros for each function provided.

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b. List 5 possible sets of functions from the list that multiply to build a quartic function.

2. Complete each statement with *always*, *sometimes*, or *never*. Explain your reasoning.

a. A quartic function _______________ has 4 real roots.

b. A function of the nth degree _______________ has n roots.

c. The number of x-intercepts _______________ matches the number of roots of a function.

d. A function _______________ has imaginary roots.

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e. A function _______________ has an odd number of imaginary roots.

3. Analyze the table shown. The function $h(x)$ is the product of $f(x)$ and $g(x)$.

x	$f(x)$	$g(x)$	$h(x) = f(x) \cdot g(x)$
−2	8	4	32
−1	5	1	5
0	4	0	0
1	5	1	5
2	8	4	32
3	13	9	117

a. Determine whether $h(x)$ is a quartic function. Explain your reasoning.

b. Determine the number of real and imaginary zeros of $h(x)$. Explain your reasoning.

c. Describe the end behavior of $h(x)$. How does this help you determine whether the function is quartic or not?

4. Analyze the table shown. The function $m(x)$ is the product of $j(x)$ and $k(x)$.

x	$j(x)$	$k(x)$	$m(x) = j(x) \cdot k(x)$
−2	4	−1	−4
−1	0	0	0
0	−2	1	−2
1	−2	2	−4
2	0	3	0
3	4	4	16

a. Determine whether $m(x)$ is a quartic function. Explain your reasoning.

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b. Determine the number of real and imaginary zeros for the function $m(x)$. Explain your reasoning.

c. Describe the end behavior of $m(x)$. How does this help you determine whether the function is quartic or not?

5. Gavin explains the relationship between the imaginary zeros of a polynomial function and the table of values for that function. Henry disagrees.

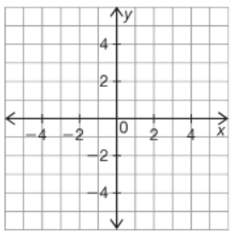
> **Gavin**
>
> A polynomial function with imaginary zeros has imaginary numbers in the table of values. For example, the function $x^2 + 4$ has 2 imaginary zeros. These values appear in the table.

> **Henry**
>
> *It is impossible for a polynomial function to have imaginary numbers in the table of values. A real input value must have a real output value.*

Who is correct? Explain your reasoning.

6. Sketch a set of functions whose product builds a quartic function with the given characteristics. Explain your reasoning. Determine similarities and differences between your graphs and your classmates' graphs.

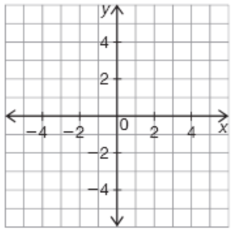
a. two imaginary roots and a double root



Explanation:

Similarities/Differences:

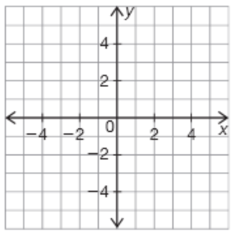
b. four distinct roots and a *y*-intercept of (0, −24)



Explanation:

Similarities/Differences:

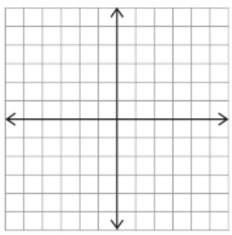
c. located in Quadrants III and IV only



Explanation:

Similarities/Differences:

d. located in quadrants II and IV only



Explanation:

Similarities/Differences:

7. What function types can be multiplied together to build a new function of degree 5? How many total zeros will the function have? How many can be imaginary?

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8. Explain the possible ways to build a function of degree n?

Be prepared to share your solutions and methods.